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Abstract

In this paper we study the departure processes of two rate-control throttles: the token bank and the leaky bucket. Using sample path methods and the notion of majorization, we analyze the effect that parameters such as the token buffer capacity and token generation period have on the vector of interdeparture times. In the transient case, we establish the monotonicity of the burst reduction in the sense of the majorization. In the case that the departure process converges in coupling to a stationary and ergodic sequence, the transient comparison results allow us to establish the monotonicity of the stationary interdeparture times in the sense of the convex ordering. Comparisons between the two flow control schemes are also established when appropriate.

Keywords: ATM networks, flow control, leaky bucket, token bank, departure process, majorization, convex ordering.

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Propriétés de lissage de trafic des mécanismes de contrôle de taux : processus de départs

Résumé

Nous étudions les processus de départs de deux mécanismes de contrôle de taux : *token bank* et *leaky bucket*. Utilisant des techniques d'analyse trajectorielle et la notion de majoration, nous analysons les effets de modification des paramètres, tel que la taille du tampon de jetons et la fréquence de génération des jetons, sur le vecteur des durées d'interdéparts. Dans le régime transitoire, nous établissons la monotonie de lissage de trafic dans le sens de majoration. Dans le cas où le processus de départs converge en couplage vers un processus stationnaire et ergodique, nous obtenons la monotonie de la durée des inter-départs stationnaire dans le sens de l'ordre convexe. La comparaison entre les deux mécanismes de contrôle est aussi effectuée.

Mots-clés: Réseaux ATM, contrôle de flue, leaky bucket, token bank, processus de départs, majoration, ordre convexe.

1 Introduction

Rate control throttles in ATM networks have received considerable attention in recent years. This is because they are open loop control schemes, are easy to implement, and have been shown through numerous performance studies to effectively reduce the burstiness in the traffic offered by a source to a network (see [12, 11, 3] for examples of such studies). The goal of this paper is to investigate the burst reduction properties of two rate-control throttles, the token bank and leaky bucket. Specifically, we study the effect that different parameters have on the departure processes produced by these two schemes.

Briefly, a rate-control throttle is associated with each data source. It generates tokens periodically and each packet emanating from the data source is required to pair up with a token prior to departing. In the event that there is no token present at the time that a packet arrives, the packet enters a data buffer. Similarly, there is a token buffer associated with the token generator. The token bank and leaky bucket schemes differ from each other only when the token buffer is full. In this case, the token bank continues to generate tokens (which are thrown out so long as the token buffer remains full) whereas the leaky bucket halts token generation until the token buffer becomes non-full.

For both of these schemes, we examine the effect of varying the token generation rate and the token buffer capacity on the departure process from the flow control mechanisms (which is the process of packet arrivals to the network). Using sample path arguments, we show that the interdeparture vector under one scheme is either majorized or weakly supermajorized by the interdeparture time vector of the same scheme with a larger token buffer capacity. We establish a similar result when we increase the length of the token generation period. These results yield convex and decreasing convex orderings between the stationary interdeparture times when these quantities exist. We also compare the departure processes of the token bank and leaky bucket control schemes when their token buffer capacities are either identical or differ by one. This result is established only in the case that the data buffer capacities are infinite.

Our work is related to the works of Kuang [7, 8], Budka [5], and Budka and Yao [6]. Kuang [7] first showed that the token bank reduces bursts in the packet arrival process to the network in the sense that the interdeparture time vector from a token bank is majorized by the interarrival time vector of the source. In [8], he presented (without detailed proofs) several monotonicity properties of the token bank. Budka and Yao [5, 6] examined monotonicity and concavity properties of the throughput of several rate-control throttles including the token bank and leaky bucket using sample path arguments. However, they did not address the burst reduction properties of these schemes. Our use of the terms token bank and leaky bucket is taken from [5]. We warn the reader that most other papers use the term "leaky bucket" when referring to the mechanism that we (and Budka and Yao) refer to as the token bank.

There are other ways to compare burst reduction properties of different mechanisms. For example Anantharam and Konstantopoulos [1] considered a token bank feeding a downstream infinite capacity queue with deterministic single server. They studied the effect that varying the token buffer capacity has on the buffer occupancy of the downstream queue. They showed that the stationary buffer occupancy at this queue is a stochastically increasing function of the token buffer size in the token bank. In a companion paper [9], we considered the effect of varying parameters on the waiting times of individual packets at such a downstream queue when it is infinite in capacity, and on the process describing the number of losses when it is finite. The results of both studies corroborate the results presented in this paper. Last, Berger and Whitt, [4] used similar arguments to study the effect of buffer allocation between the token bank data buffer and the downstream buffer.

The remainder of this paper is organized as follows. Section 2 defines and introduces a formal model for the token bank and leaky bucket. In addition, the comparison techniques and some elementary sample path properties used in the remainder of the paper are defined and derived in this section. Results established for systems with infinite capacity and finite capacity data buffers are established respectively in Sections 3 and 4. A summary is provided in Section 5.

2 Notation and Preliminaries

2.1 Model of Rate Control Throttle

Two types of rate-control throttles will be considered: the token bank (TB) and the leaky bucket (LB). In both schemes, there is a data buffer of size $0 \le B_D \le \infty$ and a token buffer of size $0 \le B_T \le \infty$. Let $B = B_D + B_T$ be the total buffer size. In order to avoid triviality, we assume that $B \ge 1$.

The two schemes differ slightly in the way that tokens are generated. In the token bank, tokens are generated periodically with constant rate $T^{-1} < \infty$ (or token generation period length $T > 0$). A generated token is accepted by the token buffer if there is empty space, i.e., if the token buffer is not full. Tokens that find the token buffer full at the times of their arrival are rejected.

In the leaky bucket, the tokens are generated periodically with constant token generation rate $T^{-1} < \infty$. When the token buffer is full, the token generator is shut off. The token generator is turned on again when the token buffer has space for at least one token. Note that, if at time t the queue length of the token buffer drops from B_T to $B_T - 1$, the next token arrival occurs at time $t + T$.

When a cell (or fixed length packet) arrives, it is accepted by the data buffer if it not full. A cell that finds the data buffer full at the time of its arrival is rejected (or marked and transmitted to the downstream system with lower priority). A cell leaves the data buffer and is transmitted to the downstream system if there is a token in the token buffer. When a cell leaves the data buffer, it consumes one token, i.e., a token leaves the token buffer at the same time. By convention, we will assume that when a token and a cell arrive simultaneously in the system, both the cell and the token are accepted, whatever the status of the data buffer and the token buffer may be. In such a case, a cell and a token leave the system at the same time.

When $B_T = 0$, the leaky bucket scheme is not defined. We assume, by convention, that there is a token buffer which is always considered to be empty. Under such an assumption, the leaky bucket behaves exactly in the same way as the token bank scheme does.

Note that in practice, there is no need for a buffer to store tokens. A token counter suffices. The token buffer serves to visualize the way how these control mechanisms work.

Unless otherwise stated, we will assume throughout this paper that the token buffer is full (so that the data buffer is empty) at the beginning.

We define following notation:

- a_n: arrival time of the n-th cell; $a_1 > 0$; for notational simplicity and without loss of generality, we assume that there is at most one cell arrival at any time;
- $\alpha_n = a_n - a_{n-1}$: n-th inter-arrival time; $\alpha_1 = a_1$;
- $\hat{a}_n$: arrival time of the n-th accepted cell; $\hat{a}_1 = a_1$;

- g_n: time epoch when the n-th token is generated; by convention, we assume $g_n = 0$, $1 \leq n \leq B_T$, and $g_{B_T+1} > 0$;
- s_n: the arrival time of the n-th accepted token; by convention, $s_n = 0$, $1 \leq n \leq B_T$, and $s_{B_T+1} > 0$;
- d_n: time epoch of the n-th cell departure;
- $\delta_n = d_n - d_{n-1}$: n-th cell inter-departure time; $\delta_1 = d_1$.

Let $A = \{a_n\}_{n=1}^{\infty}$ be the cell arrival sequence, and $\widehat{A} = \{\hat{a}_n\}_{n=1}^{\infty}$ be the arrival sequence of accepted cells. Denote by $V = \{v_n\}_{n=1}^{\infty}$ the indices of accepted cells, viz., $\hat{a}_n = a_{v_n}$. Note that when the data buffer is infinite, the sequences A and $\widehat{A}$ coincide, and $v_n = n$ for all $n = 1, 2, \cdots$.

Let $G = \{g_n\}_{n=1}^{\infty}$ be the sequence of token generation times, and $S = \{s_n\}_{n=1}^{\infty}$ the arrival time sequence of accepted tokens. Note that in the leaky bucket scheme, these two sequences are identical.

Let $\boldsymbol{\alpha} = \{\alpha_n\}_{n=1}^{\infty}$ and $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ be the sequence of interarrival times and the sequence of inter-departure times. For all $n \geq 1$, denote $\boldsymbol{\alpha}_n = (\alpha_1, \cdots, \alpha_n)$, $\boldsymbol{\delta}_n = (\delta_1, \cdots, \delta_n)$.

Define K as the set of indices of accepted cells which are instantaneously transmitted to the downstream system: $K = \{n | n \in I\!N_+, \ \hat{a}_n = d_n\}$, where $I\!N_+$ is the set of strictly positive integers. Let $\overline{K} = I\!N_+ - K$.

The following processes will also be of use in the paper.

- Q_t^D: the number of cells waiting in the data buffer at time $t \geq 0$;
- Q_t^T: the number of tokens waiting in the token buffer at time $t \geq 0$;
- $Z_t = Q_t^D - Q_t^T$: the difference between the number of cells and the number of tokens waiting in the system at time $t \geq 0$;
- D_t: the number of departures by time $t \geq 0$.

These processes are assumed to be right-continuous. Thus, $Q_{\hat{a}_n}^D$ and $Q_{\hat{a}_n}^T$ represent the numbers of cells and tokens, respectively, waiting in the system just after the arrival of the n-th accepted cell.

The above quantities will be parameterized, when necessary, by: (1) the type of control scheme, TB (for token bank) or LB (for leaky bucket); (2) the size of the data buffer; (3) the size of the token buffer; and (4) the length of the token generation period. For example, $d_n(TB, B_D, B_T, T)$ (resp. $d_n(LB, B_D, B_T, T)$) denotes the n-th departure time of the token bank (resp. leaky bucket) scheme with data buffer size B_D, token buffer size B_T and token generation period length T.

2.2 Stationarity

A sequence of real random variables $\{X_n\}_{n=1}^{\infty}$ is said to converge in coupling if there exists a stationary and ergodic sequence $\{X_n^0\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} P\{X_k = X_k^0, \ \ \forall k \geq n \} = 1.$$

Let X^0 be the generic random variable of the sequence $\{X_n^0\}_{n=1}^{\infty}$. We call X^0 the limit random variable of the sequence $\{X_n\}_{n=1}^{\infty}$. It is easily seen the individual ergodic theorem is satisfied, i.e., for all measurable functions $f : I\!R \to I\!R$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = E[f(X^0)] \quad a.s. \tag{2.1}$$

Throughout this paper we will assume that the sequence of interarrival times converges in coupling, although the transient comparison results are valid without this condition. Let α be the limit random variable and $\lambda = 1/E[\alpha]$ the arrival rate, i.e.,

$$\lambda = \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=2}^{n+1} \alpha_i \right)^{-1} \qquad a.s. \tag{2.2}$$

The assumption of convergence in coupling includes most of the models of bursty traffic used in the literature. It contains the class of stationary and ergodic sequences, which in turn includes renewal processes, Markov modulated Poisson processes, etc.

2.3 Majorization and Stochastic Ordering

The comparison and monotonicity results presented in this paper will be based on the notions of majorization and convex ordering.

Define first the notion of majorization. Let $\boldsymbol{x} = (x_1, \cdots, x_n), \boldsymbol{y} = (y_1, \cdots, y_n) \in I\!R^n$ be two real vectors. Vector $\boldsymbol{x}$ is said to be majorized by vector $\boldsymbol{y}$ (written $\boldsymbol{x} \prec \boldsymbol{y}$) if and only if

$$\begin{aligned} \sum_{i=1}^{k} x_{[i]} &\leq \sum_{i=1}^{k} y_{[i]}, \qquad k = 1, \cdots, n-1, \\ \sum_{i=1}^{n} x_{[i]} &= \sum_{i=1}^{n} y_{[i]}, \end{aligned}$$

where the notation $x_{[i]}$ is taken to be the i-th largest element of $\boldsymbol{x}$. If

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \qquad k = 1, \cdots, n,$$

then vector $\boldsymbol{x}$ is said to be weakly submajorized by vector $\boldsymbol{y}$, written $\boldsymbol{x} \prec_w \boldsymbol{y}$. If

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, \qquad k = 1, \cdots, n,$$

where the notation $x_{(i)}$ is taken to be the i-th smallest element of $\boldsymbol{x}$, then vector $\boldsymbol{x}$ is said to be weakly supermajorized by vector $\boldsymbol{y}$, written $\boldsymbol{x} \prec^w \boldsymbol{y}$.

Clearly, if $\boldsymbol{x} \prec \boldsymbol{y}$, then $\boldsymbol{x} \prec_w \boldsymbol{y}$ and $\boldsymbol{x} \prec^w \boldsymbol{y}$.

Various properties concerning these majorizations can be found in Marshall and Olkin [10]. In particular, we have the following characterizations of (weak) majorizations:

Lemma 2.1 ([10, pp. 108-109, Propositions B.1 and B.2]) *For any vectors $\boldsymbol{x}, \boldsymbol{y} \in I\!R^n$,*

- $\boldsymbol{x} \prec \boldsymbol{y}$ *if and only if* $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$ *for all convex functions* $f : I\!R \to I\!R$;
- $\boldsymbol{x} \prec_w \boldsymbol{y}$ *if and only if* $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$ *for all increasing and convex functions* $f : I\!R \to I\!R$;
- $\boldsymbol{x} \prec^w \boldsymbol{y}$ *if and only if* $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$ *for all decreasing and convex functions* $f : I\!R \to I\!R$.

Another property that will be used is the closure property of majorizations under concatenation:

Lemma 2.2 ([10, Proposition 5.A.7, p.121]) *For any vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, and $\boldsymbol{x}', \boldsymbol{y}' \in \mathbb{R}^m$, if*

$$\boldsymbol{x} \prec \text{ (resp. } \prec_w, \prec^w \text{) } \boldsymbol{y}, \qquad \text{and} \qquad \boldsymbol{x}' \prec \text{ (resp. } \prec_w, \prec^w \text{) } \boldsymbol{y}',$$

then

$$(\boldsymbol{x}, \boldsymbol{x}') \prec \text{ (resp. } \prec_w, \prec^w \text{) } (\boldsymbol{y}, \boldsymbol{y}').$$

The following lemma is easily verified by the definition of majorization.

Lemma 2.3 *Let $n \geq 1$. The vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ is weakly supermajorized by $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, $\boldsymbol{x} \prec^w \boldsymbol{y}$, if*

- $x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n$, *and*
- *for all* $1 \leq i \leq n$, $\sum_{j=1}^{i} x_j \geq \sum_{j=1}^{i} y_j$.

If moreover $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$, then $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$, $\boldsymbol{x} \prec \boldsymbol{y}$.

Define now the stochastic orderings. Let $X, Y \in \mathbb{R}$ be two random variables. The random variable X is stochastically less than the random variable Y in the sense of strong stochastic ordering ($X \leq_{st} Y$), convex ordering ($X \leq_{cx} Y$), increasing convex ordering ($X \leq_{icx} Y$), and decreasing convex ordering ($X \leq_{dcx} Y$), respectively, if

$$\begin{aligned}
E[f(X)] &\leq E[f(Y)], && \forall \text{ increasing } f : \mathbb{R}^n \to \mathbb{R}, \\
E[f(X)] &\leq E[f(Y)], && \forall \text{ convex } f : \mathbb{R}^n \to \mathbb{R}, \\
E[f(X)] &\leq E[f(Y)], && \forall \text{ increasing and convex } f : \mathbb{R}^n \to \mathbb{R}, \\
E[f(X)] &\leq E[f(Y)], && \forall \text{ decreasing and convex } f : \mathbb{R}^n \to \mathbb{R},
\end{aligned}$$

respectively, provided the expectations exist.

The reader is referred to [14] for properties concerning these orderings. It can be shown (cf. e.g. [2]) that the following equivalences hold:

Lemma 2.4 *Let $X, Y \in \mathbb{R}$ be two random variables. Then*

- $X \leq_{cx} Y$ *holds if and only if* $X \leq_{icx} Y$ *and* $EX = EY$;
- $X \leq_{cx} Y$ *holds if and only if* $X \leq_{dcx} Y$ *and* $EX = EY$.

Note that for positive random variables, the class of increasing and convex functions used in the definition of $\leq_{icx}$ ordering can be reduced to positive valued functions:

Lemma 2.5 *Let* $X, Y \in I\!R^+$ *be two positive random variables. Then* $X \leq_{icx} Y$ *if and only if the inequality* $E[f(X)] \leq E[f(Y)]$ *holds for all increasing and convex functions* $f : I\!R^+ \to I\!R^+$ *such that* $f(0) = 0$ *and* $f(x) \geq 0$, $x \geq 0$, *provided the expectations exist.*

Throughout this paper, positivity and increasingness are understood to be in the non-strict sense, i.e., "increasing" means "nondecreasing", "positive" means "nonnegative".

2.4 Preliminary Sample Path Comparison Results

We now present some preliminary comparison relations which will be used in the proofs of our main results.

Lemma 2.6 *Consider two rate-control throttles* C *and* C', *which have the same arrival sequence of accepted cells* $\widehat{A}$, *but different arrival sequences of accepted tokens* $S = \{s_n\}_{n=1}^{\infty}$ *and* $S' = \{s'_n\}_{n=1}^{\infty}$ *for* C *and* C', *respectively. Let* K *and* K' *be the sets of instantaneous departure points of* C *and* C', *respectively. If for all* $n \geq 1$, $s_n \geq s'_n$, *then* $K \subseteq K'$.

Proof. Note that $n \in K$ if and only if $\hat{a}_n \geq s_n$. Thus, for all $n \in K$, $\hat{a}_n \geq s_n \geq s'_n$, so that $n \in K'$. ∎

The following lemma was shown in [9]:

Lemma 2.7 ([9]) *Let* C *be a rate-control throttle with token generation period length* T, *token buffer size* $B_T \geq 1$ *and the arrival sequence of accepted cells* $\widehat{A}$. *Let* $K = \{n_1, n_2, \cdots, n_k, \cdots, n_{k_0}\}$, *where* $k_0 \leq \infty$, $1 = n_1 < n_2 < \cdots < n_k < \cdots < n_{k_0}$ *(by convention,* $n_{k_0} = \infty$ *if* $k_0 = \infty$*). Then, for all* $k \geq 2$,

$$d_{n_k} - d_{n_{k-1}} = \hat{a}_{n_k} - \hat{a}_{n_{k-1}}. \tag{2.3}$$

Moreover, for all $2 \leq k \leq k_0$*, if* $n_k > n_{k-1} + 1$*, then*

$$\delta_{n_{k-1}+1} \leq T, \quad \delta_{n_{k-1}+2} = \cdots = \delta_{n_k-1} = T, \quad \delta_{n_k} \geq T. \tag{2.4}$$

Further, if $k_0 < \infty$*, then*

$$\delta_{n_{k_0}+1} \leq T, \qquad \delta_i = T, \quad i \geq n_{k_0} + 2. \tag{2.5}$$

The above lemmas yield the following basic comparison result.

Lemma 2.8 *Consider two rate-control throttles* C *and* C'*, which have token buffer sizes* $B_T \geq 1$ *and* $B'_T \geq 0$*, respectively, token generation period lengths* T *and* T'*, respectively, arrival sequences of accepted tokens* $S = \{s_n\}_{n=1}^{\infty}$ *and* $S' = \{s'_n\}_{n=1}^{\infty}$*, respectively, and the common arrival sequence of accepted cells* $\widehat{A}$*. Let* K *(resp.* K'*) be the instantaneous departure points of* C *(resp.* C'*), and* $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ *(resp.* $\boldsymbol{\delta}' = \{\delta'_n\}_{n=1}^{\infty}$*) be the sequence of inter-departure times of* C *(resp.* C'*). If for all* $n \geq 1$*,* $s_n \geq s'_n$*, then*

$$\boldsymbol{\delta}_n \prec \boldsymbol{\delta}'_n, \quad n \in K, \quad \textit{and} \quad \boldsymbol{\delta}_n \prec^w \boldsymbol{\delta}'_n, \quad n \in \overline{K}.$$

Proof. Applying first Lemma 2.6 implies that $K \subseteq K'$. Let $K = \{n_1, n_2, \cdots, n_k, \cdots, n_{k_0}\}$, where $1 = n_1 < n_2 < \cdots < n_k < \cdots < n_{k_0} \leq \infty$.

Let d_n (resp. d'_n) be the n-th departure time in C (resp. C'). It then follows that for all $n = 1, 2, \cdots$,

$$d_n = \max(\hat{a}_n, s_n) \geq \max(\hat{a}_n, s'_n) = d'_n.$$

Observe that for all $k = 1, 2, \cdots, k_0$, $d_{n_k} = d'_{n_k} = \hat{a}_{n_k}$. Thus, appealing to Lemmas 2.7 and 2.3 implies that for all $k = 2, 3, \cdots, k_0$,

$$(\delta_{n_{k-1}+1}, \delta_{n_{k-1}+2}, \cdots, \delta_{n_k}) \prec \left(\delta'_{n_{k-1}+1}, \delta'_{n_{k-1}+2}, \cdots, \delta'_{n_k}\right).$$

Using the closure property of majorization under concatenation (cf. Lemma 2.2) entails that for all $k = 1, 2, \cdots, k_0$,

$$(\delta_1, \delta_2, \cdots, \delta_{n_k}) \prec \left(\delta'_1, \delta'_2, \cdots, \delta'_{n_k}\right).$$

Lemmas 2.7 and 2.3 imply

$$(\delta_{n_k+1}, \delta_{n_k+2}, \cdots, \delta_n) \prec^w \left(\delta'_{n_k+1}, \delta'_{n_k+2}, \cdots, \delta'_n\right)$$

for $n_k < n < n_{k+1}$, $k = 1, 2, \ldots, k_0$. Since

$$(\delta_1, \delta_2, \cdots, \delta_{n_k}) \prec \left(\delta'_1, \delta'_2, \cdots, \delta'_{n_k}\right),$$

we obtain (cf. Lemma 2.2) that for all $n \in \overline{K}$,

$$(\delta_1, \delta_2, \cdots, \delta_n) \prec^w (\delta'_1, \delta'_2, \cdots, \delta'_n).$$

■

3 Main Results in the Infinite Data Buffer Case

We consider in this section the case where the data buffer has infinite capacity, $B_D = \infty$. In this case, all cells are accepted, $a_n = \hat{a}_n$.

3.1 Comparison in Transient Regime

We first present the following inequalities which come from [9].

Lemma 3.1 **([9])** *Assume* $B_T \geq 1$. *Then for any cell arrival sequence* $\boldsymbol{\alpha}$,

$$\begin{aligned}
s_n(TB, \infty, B_T, T) &\leq s_n(LB, \infty, B_T, T) && && (3.1)\\
s_n(LB, \infty, B_T + 1, T) &\leq s_n(TB, \infty, B_T, T) && && (3.2)\\
s_n(TB, \infty, B_T, T) &\leq s_n(TB, \infty, B_T, T'), && T' = mT, \quad m \in I\!N_+ && (3.3)\\
s_n(LB, \infty, B_T, T) &\leq s_n(LB, \infty, B_T, T'), && T' \geq T && (3.4)
\end{aligned}$$

The following theorem compares the leaky bucket scheme with the token bank scheme.

Theorem 3.1 *Assume $B_T \geq 1$. Then for any fixed cell arrival sequence A,*

$$\begin{aligned} \boldsymbol{\delta}_n(LB,\infty,B_T,T) &\prec \boldsymbol{\delta}_n(TB,\infty,B_T,T), && \forall n \in K(LB,\infty,B_T,T), \\ \boldsymbol{\delta}_n(LB,\infty,B_T,T) &\prec^w \boldsymbol{\delta}_n(TB,\infty,B_T,T), && \forall n \in \overline{K}(LB,\infty,B_T,T), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \boldsymbol{\delta}_n(TB,\infty,B_T,T) &\prec \boldsymbol{\delta}_n(LB,\infty,B_T+1,T), && \forall n \in K(TB,\infty,B_T,T), \\ \boldsymbol{\delta}_n(TB,\infty,B_T,T) &\prec^w \boldsymbol{\delta}_n(LB,\infty,B_T+1,T), && \forall n \in \overline{K}(TB,\infty,B_T,T). \end{aligned} \tag{3.6}$$

Proof. Relation (3.5) is a consequence of Lemma 2.8 and inequality (3.1), whereas relation (3.6) follows from Lemma 2.8 and inequality (3.2). ∎

Theorem 3.1 implies the following monotonicity of the variability in the rate-control throttles with respect to the token buffer size.

Corollary 3.1 *Assume $B_T \geq 1$. Then for any fixed cell arrival sequence A,*

$$\begin{aligned} \boldsymbol{\delta}_n(TB,\infty,B_T,T) &\prec \boldsymbol{\delta}_n(TB,\infty,B_T+1,T), && \forall n \in K(TB,\infty,B_T,T), \\ \boldsymbol{\delta}_n(TB,\infty,B_T,T) &\prec^w \boldsymbol{\delta}_n(TB,\infty,B_T+1,T), && \forall n \in \overline{K}(TB,\infty,B_T,T), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \boldsymbol{\delta}_n(LB,\infty,B_T,T) &\prec \boldsymbol{\delta}_n(LB,\infty,B_T+1,T), && \forall n \in K(LB,\infty,B_T,T), \\ \boldsymbol{\delta}_n(LB,\infty,B_T,T) &\prec^w \boldsymbol{\delta}_n(LB,\infty,B_T+1,T), && \forall n \in \overline{K}(LB,\infty,B_T,T). \end{aligned} \tag{3.8}$$

When the token buffer has infinite capacity, the inter-departure times are identical to the inter-arrival times. Therefore, the leaky bucket and the token bank flow control schemes reduce the burstiness:

Corollary 3.2 *Assume $B_T \geq 1$. Then for any fixed cell arrival sequence A,*

$$\begin{aligned} \boldsymbol{\delta}_n(TB,\infty,B_T,T) &\prec \boldsymbol{\alpha}_n, && \forall n \in K(TB,\infty,B_T,T), \\ \boldsymbol{\delta}_n(TB,\infty,B_T,T) &\prec^w \boldsymbol{\alpha}_n, && \forall n \in \overline{K}(TB,\infty,B_T,T), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \boldsymbol{\delta}_n(LB,\infty,B_T,T) &\prec \boldsymbol{\alpha}_n, && \forall n \in K(LB,\infty,B_T,T), \\ \boldsymbol{\delta}_n(LB,\infty,B_T,T) &\prec^w \boldsymbol{\alpha}_n, && \forall n \in \overline{K}(LB,\infty,B_T,T). \end{aligned} \tag{3.10}$$

Remark: Relations (3.7) and (3.9) were presented in [8] and [7], respectively. Note, however, that these relations do not hold when $B_T = 0$ (see the remark at the end of this section).

We now consider the effects of the token generation period length on the burst reduction of the rate-control throttles.

Theorem 3.2 *Assume $B_T \geq 1$. For any fixed cell arrival sequence A, if $T = mT'$, where m is an arbitrary strictly positive integer, then*

$$\begin{array}{llll} \delta_n(TB,\infty,B_T,T) & \prec & \delta_n(TB,\infty,B_T,T'), & n \in K(TB,\infty,B_T,T), \\ \delta_n(TB,\infty,B_T,T) & \prec^w & \delta_n(TB,\infty,B_T,T'), & n \in \overline{K}(TB,\infty,B_T,T). \end{array} \tag{3.11}$$

Proof. Lemma 2.8 together with relation (3.3) imply the assertion of the theorem. ∎

Remark: Note that the above comparison result holds only when T is an integer multiple of T'. In fact, for any $B_T \geq 1$ and any $T > T'$, such that T/T' is not an integer, one can construct an arrival sequence of cells for which $K(TB,\infty,B_T,T) = K(TB,\infty,B_T,T')$ are infinite sets, and

$$\delta_n(TB,\infty,B_T,T') \prec \delta_n(TB,\infty,B_T,T), \quad n \in K(TB,\infty,B_T,T).$$

For example, let $B_T = 1, T = 3$ and $T' = 2$. Define the cell arrival times as $A = \{2.5, 3, 8.5, 9, \cdots, 6n+2.5, 6n+3, \cdots\}$. A simple calculation yields

$$\begin{array}{lll} \delta(TB,\infty,1,3) & = & \{2.5, 0.5, 5.5, 0.5, 5.5, \cdots\} \\ \delta(TB,\infty,1,2) & = & \{2.5, 1.5, 4.5, 1.5, 4.5, \cdots\} \end{array}$$

However, it is possible to show (cf. [8]) that if $B_T < B'_T$ and $T \geq T'$, then

$$\begin{array}{llll} \delta_n(TB,\infty,B_T,T) & \prec & \delta_n(TB,\infty,B'_T,T'), & n \in K(TB,\infty,B_T,T), \\ \delta_n(TB,\infty,B_T,T) & \prec^w & \delta_n(TB,\infty,B'_T,T'), & n \in \overline{K}(TB,\infty,B_T,T). \end{array}$$

Theorem 3.3 *Assume $B_T \geq 1$. For any fixed cell arrival sequence A, if $T \geq T'$, then*

$$\begin{array}{llll} \delta_n(LB,\infty,B_T,T) & \prec & \delta_n(LB,\infty,B_T,T'), & n \in K(LB,\infty,B_T,T), \\ \delta_n(LB,\infty,B_T,T) & \prec^w & \delta_n(LB,\infty,B_T,T'), & n \in \overline{K}(LB,\infty,B_T,T). \end{array} \tag{3.12}$$

Proof. Lemma 2.8 together with relation (3.4) imply the assertion of the theorem. ∎

3.2 Comparison in the Stationary Regime

We now consider the comparison in the stationary regime. We will therefore assume that the sequences of interdeparture times under analysis converge in coupling. This is trivially the case when $\lambda T > 1$. If the sequence of cell interarrival times is a stationary and ergodic, and $\lambda T < 1$, then the sequence of the interdeparture times converges in coupling. The interested reader is referred to [9] for detailed discussions.

Theorem 3.4 *Assume $B_T \geq 1$. If the sequences $\boldsymbol{\delta}(LB, \infty, B_T, T)$, $\boldsymbol{\delta}(TB, \infty, B_T, T)$ and $\boldsymbol{\delta}(LB, \infty, B_T + 1, T)$ converge in coupling with limit variables $\delta(LB, \infty, B_T, T)$, $\delta(TB, \infty, B_T, T)$ and $\delta(LB, \infty, B_T + 1, T)$, respectively, then*

$$\delta(LB, \infty, B_T, T) \leq_{cx} \delta(TB, \infty, B_T, T) \leq_{cx} \delta(LB, \infty, B_T + 1, T). \tag{3.13}$$

Proof. It follows from Theorem 3.1 that, for all $k = 1, 2, \cdots$,

$$\boldsymbol{\delta}_k(LB, \infty, B_T, T) \prec^w \boldsymbol{\delta}_k(TB, \infty, B_T, T) \prec^w \boldsymbol{\delta}_k(LB, \infty, B_T + 1, T).$$

Owing to the characterization of majorization (cf. Lemma 2.1), we obtain that for all decreasing and convex functions $f : I\!R \to I\!R$,

$$\sum_{i=1}^{k} f\left(\delta_i(LB, \infty, B_T, T)\right) \leq \sum_{i=1}^{k} f\left(\delta_i(TB, \infty, B_T, T)\right) \leq \sum_{i=1}^{k} f\left(\delta_i(LB, \infty, B_T + 1, T)\right).$$

Dividing each term by k in the above inequalities and letting k tend to infinity implies (cf. (2.1)) that

$$E[f(\delta(LB, \infty, B_T, T))] \leq E[f(\delta(TB, \infty, B_T, T))] \leq E[f(\delta(LB, \infty, B_T + 1, T))].$$

Therefore,

$$\delta(LB, \infty, B_T, T) \leq_{dcx} \delta(TB, \infty, B_T, T) \leq_{dcx} \delta(LB, \infty, B_T + 1, T). \tag{3.14}$$

Since the data buffer is infinite, we have that

$$E[\delta(LB, \infty, B_T, T)] = E[\delta(TB, \infty, B_T, T)] = E[\delta(LB, \infty, B_T + 1, T)] = \max(1/\lambda, T). \tag{3.15}$$

Relations (3.14) and (3.15), together with Lemma 2.4, readily entail (3.13). ∎

As a consequence of the above theorem, we obtain the monotonicity of the variability of the inter-departure times with respect to the token buffer size:

Corollary 3.3 *Assume $B_T \geq 1$. If the sequences $\boldsymbol{\delta}(TB, \infty, B_T, T)$, $\boldsymbol{\delta}(TB, \infty, B_T+1, T)$, $\boldsymbol{\delta}(LB, \infty, B_T, T)$ and $\boldsymbol{\delta}(LB, \infty, B_T+1, T)$ converge in coupling with limit variables $\delta(TB, \infty, B_T, T)$, $\delta(TB, \infty, B_T+1, T)$, $\delta(LB, \infty, B_T, T)$ and $\delta(LB, \infty, B_T+1, T)$, respectively, then*

$$\delta(TB, \infty, B_T, T) \quad \leq_{cx} \quad \delta(TB, \infty, B_T+1, T), \tag{3.16}$$

$$\delta(LB, \infty, B_T, T) \quad \leq_{cx} \quad \delta(LB, \infty, B_T+1, T). \tag{3.17}$$

Using the same arguments as in the proof of Theorem 3.4, along with the results of Corollary 3.2, Theorems 3.2 and 3.3, we can show the following (decreasing) convex orderings:

Theorem 3.5 *Assume $B_T \geq 1$. If the sequences $\boldsymbol{\delta}(TB, \infty, B_T, T)$ and $\boldsymbol{\delta}(LB, \infty, B_T, T)$ converge in coupling with limit variables $\delta(TB, \infty, B_T, T)$ and $\delta(LB, \infty, B_T, T)$, respectively, then*

$$\delta(TB, \infty, B_T, T) \quad \leq_{dcx} \quad \alpha, \tag{3.18}$$

$$\delta(LB, \infty, B_T, T) \quad \leq_{dcx} \quad \alpha. \tag{3.19}$$

If moreover $\lambda T < 1$, then

$$\delta(TB, \infty, B_T, T) \quad \leq_{cx} \quad \alpha, \tag{3.20}$$

$$\delta(LB, \infty, B_T, T) \quad \leq_{cx} \quad \alpha. \tag{3.21}$$

Theorem 3.6 *Assume $B_T \geq 1$. If $T = T'm$, where $m \in I\!N_+$, and the sequences $\boldsymbol{\delta}(TB, \infty, B_T, T)$ and $\boldsymbol{\delta}(TB, \infty, B_T, T')$ converge in coupling with limit variables $\delta(TB, \infty, B_T, T)$ and $\delta(TB, \infty, B_T, T')$, respectively, then*

$$\delta(TB, \infty, B_T, T) \leq_{dcx} \delta(TB, \infty, B_T, T'). \tag{3.22}$$

If moreover $\lambda T < 1$, then

$$\delta(TB, \infty, B_T, T) \leq_{cx} \delta(TB, \infty, B_T, T'), \tag{3.23}$$

Theorem 3.7 *Assume $B_T \geq 1$. If $T' \leq T$ and the sequences $\boldsymbol{\delta}(LB, \infty, B_T, T)$ and $\boldsymbol{\delta}(LB, \infty, B_T, T')$ converge in coupling with limit variables $\delta(LB, \infty, B_T, T)$ and $\delta(LB, \infty, B_T, T')$, respectively, then*

$$\delta(LB, \infty, B_T, T) \leq_{dcx} \delta(LB, \infty, B_T, T'). \tag{3.24}$$

If moreover $\lambda T < 1$, then

$$\delta(LB, \infty, B_T, T) \leq_{cx} \delta(LB, \infty, B_T, T'). \tag{3.25}$$

Remark: Note that the results obtained in this section pertaining to the variability of interdeparture times do not hold in general when the token buffer size is zero ($B_T = 0$). The following simple example indicates that when $B_T = 0$, the departure sequence may be burstier (in the sense of majorization and convex ordering) than the arrival sequence. Let the token generation period length be $T = 3$. The interarrival times of the cells are $4, 5, 4, 5, 4, 5, \cdots$. Then, one can see that the interdeparture times are $3, 6, 3, 6, 3, 6, \cdots$.

4 Main Results in the Finite Data Buffer Case

In this section, we consider the case when the data buffer is finite: $B_D < \infty$. Our results rely heavily on the following *invariance* properties of the rate-control throttles that we have been studying. These were established in [4] for the token bank and in [9] for the leaky bucket.

Theorem 4.1 *Consider two rate-control throttles with data buffer sizes $B_D \geq 0$ and $B'_D \geq 0$, respectively, and token buffer sizes $B_T \geq 0$ and $B'_T \geq 0$, respectively. If $B_D + B_T = B'_D + B'_T$, then for any token generation period length T and any arbitrarily fixed cell arrival sequence $A = \{a_n\}_{n=1}^{\infty}$, the arrival sequences of accepted cells are identical:*

$$\widehat{A}(TB, B_D, B_T, T) = \widehat{A}(TB, B'_D, B'_T, T), \tag{4.1}$$

$$\widehat{A}(LB, B_D, B_T, T) = \widehat{A}(LB, B'_D, B'_T, T). \tag{4.2}$$

4.1 Sensitivity

Theorem 4.1 implies that the throughput is insensitive to the partition of the sizes of data buffer and token buffer. However, as is shown below, the variability in the departure process is sensitive to this partition.

Theorem 4.2 *Assume $B_D \geq 0$ and $B_T \geq 1$. Then for any fixed cell arrival sequence A,*

$$\begin{array}{llll} \delta_n(TB, B_D+1, B_T, T) & \prec & \delta_n(TB, B_D, B_T+1, T), & \forall n \in K(TB, B_D+1, B_T, T); \\ \delta_n(TB, B_D+1, B_T, T) & \prec^w & \delta_n(TB, B_D, B_T+1, T), & \forall n \in \overline{K}(TB, B_D+1, B_T, T). \end{array} \tag{4.3}$$

$$\begin{array}{llll} \delta_n(LB, B_D+1, B_T, T) & \prec & \delta_n(LB, B_D, B_T+1, T), & \forall n \in K(LB, B_D+1, B_T, T); \\ \delta_n(LB, B_D+1, B_T, T) & \prec^w & \delta_n(LB, B_D, B_T+1, T), & \forall n \in \overline{K}(LB, B_D+1, B_T, T). \end{array} \tag{4.4}$$

Proof. Owing to Theorem 4.1,

$$\begin{aligned}\widehat{A}(TB, B_D+1, B_T, T) &= \widehat{A}(TB, B_D, B_T+1, T),\\ \widehat{A}(LB, B_D+1, B_T, T) &= \widehat{A}(LB, B_D, B_T+1, T).\end{aligned}$$

Observe that an accepted cell sees the system as if it had infinite data buffer. Thus, the above two relations allow us to apply Corollary 3.1 and to obtain relations (4.3) and (4.4). ∎

Theorem 4.3 *Assume $B_D \geq 0$ and $B_T \geq 1$. If the sequences $\boldsymbol{\delta}(TB, B_D+1, B_T, T)$, $\boldsymbol{\delta}(TB, B_D, B_T+1, T)$, $\boldsymbol{\delta}(LB, B_D+1, B_T, T)$ and $\boldsymbol{\delta}(LB, B_D, B_T+1, T)$ converge in coupling with limit variables $\delta(TB, B_D+1, B_T, T)$, $\delta(TB, B_D, B_T+1, T)$, $\delta(LB, B_D+1, B_T, T)$ and $\delta(LB, B_D, B_T+1, T)$, respectively, then*

$$\delta(TB, B_D+1, B_T, T) \leq_{cx} \delta(TB, B_D, B_T+1, T), \tag{4.5}$$

$$\delta(LB, B_D+1, B_T, T) \leq_{cx} \delta(LB, B_D, B_T+1, T). \tag{4.6}$$

Proof. As in the proof of Theorem 3.4, we can show that relations (4.3) and (4.4) imply

$$\begin{aligned}\delta(TB, B_D+1, B_T, T) &\leq_{dcx} \delta(TB, B_D, B_T+1, T),\\ \delta(LB, B_D+1, B_T, T) &\leq_{dcx} \delta(LB, B_D, B_T+1, T).\end{aligned}$$

Theorem 4.1 implies that

$$\begin{aligned}E[\delta(TB, B_D+1, B_T, T)] &= E[\delta(TB, B_D, B_T+1, T)],\\ E[\delta(LB, B_D+1, B_T, T)] &= E[\delta(LB, B_D, B_T+1, T)].\end{aligned}$$

Therefore, cf. Lemma 2.4, relations (4.5) and (4.6) hold. ∎

4.2 Monotonicity

We now establish the monotonicity of the variability with respect to the data buffer size.

Theorem 4.4 *Assume $B_D \geq 0$, $B_T \geq 0$ and $B = B_D + B_T \geq 1$. Assume further $K(TB, B_D+1, B_T, T)$ and $K(LB, B_D+1, B_T, T)$ are almost surely infinite. If the sequences $\boldsymbol{\delta}(LB, B_D+$*

$1, B_T, T)$, $\boldsymbol{\delta}(LB, B_D, B_T, T)$, $\boldsymbol{\delta}(TB, B_D+1, B_T, T)$ *and* $\boldsymbol{\delta}(TB, B_D, B_T, T)$ *converge in coupling with limit variables* $\delta(LB, B_D+1, B_T, T)$, $\delta(LB, B_D, B_T, T)$, $\delta(TB, B_D+1, B_T, T)$ *and* $\delta(TB, B_D, B_T, T)$, *respectively, then*

$$\delta(TB, B_D + 1, B_T, T) \leq_{icx} \delta(TB, B_D, B_T, T), \tag{4.7}$$

$$\delta(LB, B_D + 1, B_T, T) \leq_{icx} \delta(LB, B_D, B_T, T). \tag{4.8}$$

Note that the sets $K(TB, B_D + 1, B_T, T)$ and $K(LB, B_D + 1, B_T, T)$ are almost surely infinite if either $\lambda T < 1$, or the interarrival times have infinite support and $B_D < \infty$.

In order to prove the above theorem, we need three lemmas whose proofs are provided in Appendix A.

Lemma 4.1 *Assume* $B_D \geq 0$ *and* $B_T = 0$. *Let* A *be an arbitrary cell arrival sequence. There exists a counting process* $D'_t(TB, B_D + 1, B_T, T)$, *constructed by thinning the departure counting process* $D_t(TB, B_D + 1, B_T, T)$, *such that*

$$\delta'_k(TB, B_D + 1, B_T, T) = \delta_k(TB, B_D, B_T, T), \qquad k = 1, 2, \cdots, \tag{4.9}$$

where $\{\delta'_n(TB, B_D+1, B_T, T)\}_{n=1}^{\infty}$ *is the sequence of inter-departure times associated with* $D'_t(TB, B_D+1, B_T, T)$.

Lemma 4.2 *Assume* $B_D \geq 0$ *and* $B_T \geq 1$. *Let cell arrival sequence* A *be arbitrarily fixed. If* $K(TB, B_D+1, B_T, T)$ *is an infinite set, then, there exist a counting process* $D'_t(TB, B_D+1, B_T, T)$, *constructed by thinning the departure counting process* $D_t(TB, B_D + 1, B_T, T)$, *and an infinite sequence of integers* $\{m_k\}_{k=1}^{\infty}$, $1 = m_1 < m_2 < \cdots < m_k < \cdots$, *such that*

$$\boldsymbol{\delta}'_{m_k}(TB, B_D + 1, B_T, T) \prec \boldsymbol{\delta}_{m_k}(TB, B_D, B_T, T), \qquad k = 1, 2, \cdots, \tag{4.10}$$

where $\boldsymbol{\delta}'(TB, B_D+1, B_T, T) = \{\delta'_n(TB, B_D+1, B_T, T)\}_{n=1}^{\infty}$ *is the sequence of inter-departure times associated with* $D'_t(TB, B_D + 1, B_T, T)$.

Lemma 4.3 *Assume* $B_D \geq 0$ *and* $B_T \geq 1$. *Let cell arrival sequence* A *be arbitrarily fixed. If* $K(LB, B_D+1, B_T, T)$ *is an infinite set, then, there exist a counting process* $D'_t(LB, B_D+1, B_T, T)$, *constructed by thinning the departure counting process* $D_t(LB, B_D + 1, B_T, T)$, *and an infinite sequence of integers* $\{m_k\}_{k=1}^{\infty}$, $1 = m_1 < m_2 < \cdots < m_k < \cdots$, *such that*

$$\boldsymbol{\delta}'_{m_k}(LB, B_D + 1, B_T, T) \prec \boldsymbol{\delta}_{m_k}(LB, B_D, B_T, T), \qquad k = 1, 2, \cdots, \tag{4.11}$$

where $\boldsymbol{\delta}'(LB, B_D + 1, B_T, T) = \{\delta'_n(LB, B_D + 1, B_T, T)\}_{n=1}^{\infty}$ *is the sequence of inter-departure times associated with* $D'_t(LB, B_D + 1, B_T, T)$.

Proof of Theorem 4.4.

We consider first relation (4.7) when $B_T \geq 1$. Owing to Lemma 4.2, we obtain a counting process $D'_t(TB, B_D + 1, B_T, T)$, which is a thinning of the departure process $D_t(TB, B_D + 1, B_T, T)$, and an infinite sequence of integers $\{m_k\}_{k=1}^{\infty}$, $1 = m_1 < m_2 < \cdots < m_k < \cdots$, such that the sequence of inter-departure times $\boldsymbol{\delta}'(TB, B_D + 1, B_T, T) = \{\delta'_n(TB, B_D + 1, B_T, T)\}_{n=1}^{\infty}$ associated with $D'_t(TB, B_D + 1, B_T, T)$ is majorized by $\boldsymbol{\delta}(TB, B_D, B_T, T)$:

$$\boldsymbol{\delta}'_{m_k}(TB, B_D + 1, B_T, T) \prec \boldsymbol{\delta}_{m_k}(TB, B_D, B_T, T), \qquad k = 1, 2, \cdots.$$

Thus, the characterization of majorization implies that for all increasing and convex functions $f : I\!R \to I\!R$,

$$\sum_{i=1}^{m_k} f(\delta'_i(TB, B_D + 1, B_T, T)) \leq \sum_{i=1}^{m_k} f(\delta_i(TB, B_D, B_T, T)). \tag{4.12}$$

Since the inter-departure times are positive, we can assume (cf. Lemma 2.5) without loss of generality that $f(0) = 0$ and $f(x) \geq 0$ for all $x \geq 0$.

For all $i \geq 2$, let $u_i \geq 0$ be the number of points deleted from $D_t(TB, B_D + 1, B_T, T)$ in the time interval $(d'_{i-1}(TB, B_D + 1, B_T, T), d'_i(TB, B_D + 1, B_T, T))$ while constructing the process $D'_t(TB, B_D + 1, B_T, T)$. Let $d_{l_1^i}(TB, B_D + 1, B_T, T), \cdots, d_{l_{u_i}^i}(TB, B_D + 1, B_T, T)$ be the deleted points of $D_t(TB, B_D + 1, B_T, T)$. (In fact, relation (A.7) implies that $u_i \leq 1$). It then follows that

$$\delta'_i(TB, B_D + 1, B_T, T) = \delta_{n_i}(TB, B_D + 1, B_T, T) + \sum_{j=1}^{u_i} \delta_{l_j^i}(TB, B_D + 1, B_T, T),$$

where n_i is the index of the point in $D_t(TB, B_D + 1, B_T, T)$ which corresponds to $d'_i(TB, B_D + 1, B_T, T)$ in $D'_t(TB, B_D + 1, B_T, T)$, viz., $d_{n_i}(TB, B_D + 1, B_T, T) = d'_i(TB, B_D + 1, B_T, T)$. Clearly, $n_i \geq i$.

Since f is a convex function, we have that for all $i \geq 2$,

$$f\left(\delta_i'(TB, B_D+1, B_T, T)\right) \geq f\left(\delta_{n_i}(TB, B_D+1, B_T, T)\right) + \sum_{j=1}^{u_i} f\left(\delta_{l_j^i}(TB, B_D+1, B_T, T)\right). \quad (4.13)$$

Therefore,

$$\sum_{i=1}^{n_{m_k}} f(\delta_i(TB, B_D+1, B_T, T)) \leq \sum_{i=1}^{m_k} f(\delta_i'(TB, B_D+1, B_T, T)) \leq \sum_{i=1}^{m_k} f(\delta_i(TB, B_D, B_T, T)),$$

so that, using the fact that $n_{m_k} \geq m_k$ and the assumption of convergence in coupling,

$$\begin{aligned} E[f(\delta(TB, B_D+1, B_T, T))] &= \lim_{k\to\infty} \frac{1}{n_{m_k}} \sum_{i=1}^{n_{m_k}} f(\delta_i(TB, B_D+1, B_T, T)) \\ &\leq \lim_{k\to\infty} \frac{1}{m_k} \sum_{i=1}^{m_k} f(\delta_i(TB, B_D, B_T, T)) \\ &= E[f(\delta(TB, B_D, B_T, T))]. \end{aligned}$$

Hence, relation (4.7) holds for $B_T \geq 1$. The case $B_T = 0$ and relation (4.8) can be shown in analogous ways using Lemmas 4.1 and 4.3. ∎

Remark: In the case of $B_T = 0$, the "$\leq_{icx}$" relation can be replaced with a "$\leq_{st}$" ordering in (4.7). This is because the underlying vectorial relation in Lemma 4.1 is "$=$" instead of "$\prec$".

5 Summary

In this paper we have studied the effect that the token bank and leaky bucket have on the process of packet arrivals to a network. We have shown that they both decrease burstiness in the sense of a majorization on the interdeparture time vectors. We also established monotonicity properties in the form of majorizations with respect to different parameters such as token buffer capacity and token generation period. In addition, we established comparisons between the two schemes. These results were established both in the transient regime and in the stationary regime. In the latter case, the comparisons convert to convex and decreasing convex orderings.

A Proofs of Lemmas 4.1, 4.2 and 4.3

In this appendix, we prove Lemmas 4.1, 4.2 and 4.3 which are restated below.

Lemma 4.1 *Assume $B_D \geq 0$ and $B_T = 0$. Let A be an arbitrary cell arrival sequence. There exists a counting process $D'_t(TB, B_D + 1, B_T, T)$, constructed by thinning the departure counting process $D_t(TB, B_D + 1, B_T, T)$, such that*

$$\delta'_k(TB, B_D + 1, B_T, T) = \delta_k(TB, B_D, B_T, T), \qquad k = 1, 2, \cdots, \tag{A.1}$$

where $\{\delta'_n(TB, B_D+1, B_T, T)\}_{n=1}^{\infty}$ is the sequence of inter-departure times associated with $D'_t(TB, B_D+1, B_T, T)$.

Proof. It suffices to show that $Z_t(TB, B_D, B_T, T) \leq Z_t(TB, B_D + 1, B_T, T)$. This is easily done by induction on the times of arrivals and token generation times, $0 = t_1 < t_2 < \cdots$. Clearly, it is true at $t = t_1$. Assume now that it is true for $t \leq t_i$. Clearly it holds for $t_i < t < t_{i+1}$. We will establish it for $t = t_{i+1}$. There are two cases according to whether the next event is an arrival or departure. In the case of a cell arrival

$$\begin{aligned} Z_{t_{i+1}}(TB, B_D, B_T, T) &= \min(Z_{t_i}(TB, B_D, B_T, T) + 1, B_D), \\ &\leq \min(Z_{t_i}(TB, B_D, B_T, T) + 1, B_D + 1), \\ &\leq \min(Z_{t_i}(TB, B_D + 1, B_T, T) + 1, B_D + 1), \\ &= Z_{t_{i+1}}(TB, B_D + 1, B_T, T). \end{aligned}$$

In the case of a token arrival,

$$\begin{aligned} Z_{t_{i+1}}(TB, B_D, B_T, T) &= \max(Z_{t_i}(TB, B_D, B_T, T) - 1, -B_T), \\ &\leq \max(Z_{t_i}(TB, B_D + 1, B_T, T) - 1, -B_T), \\ &= Z_{t_{i+1}}(TB, B_D + 1, B_T, T). \end{aligned}$$

which completes the proof. ∎

Lemma 4.2 *Assume $B_D \geq 0$ and $B_T \geq 1$. Let cell arrival sequence A be arbitrarily fixed. If $K(TB, B_D+1, B_T, T)$ is an infinite set, then, there exist a counting process $D'_t(TB, B_D+1, B_T, T)$, constructed by thinning the departure counting process $D_t(TB, B_D + 1, B_T, T)$, and an infinite* *sequence of integers* $\{m_k\}_{k=1}^{\infty}$, $1 = m_1 < m_2 < \cdots < m_k < \cdots$, *such that*

$$\delta'_{m_k}(TB, B_D + 1, B_T, T) \prec \delta_{m_k}(TB, B_D, B_T, T), \qquad k = 1, 2, \cdots, \tag{A.2}$$

where $\delta'(TB, B_D+1, B_T, T) = \{\delta'_n(TB, B_D+1, B_T, T)\}_{n=1}^{\infty}$ *is the sequence of inter-departure times associated with* $D'_t(TB, B_D + 1, B_T, T)$.

Proof. Note that the sequences of cell (resp. token) arrivals are the same in the two systems. Let $\{t_n\}_{n=1}^{\infty}$ be the sequence of time epochs when a cell or a token arrives in both control schemes, where $0 = t_1 < t_2 < \cdots < t_n < \cdots$. We first show by induction that for all $n = 1, 2, \cdots$,

$$0 \le Z_{t_n}(TB, B_D + 1, B_T, T) - Z_{t_n}(TB, B_D, B_T, T) \le 1. \tag{A.3}$$

Relation (A.3) trivially holds when $n = 1$. Assume that (A.3) holds for some $n \ge 1$. Consider time t_{n+1}. There are three cases.

Case 1: Both a cell and a token arrive at time t_{n+1}.

Then, in both control schemes the token and the cell are accepted and a cell is transmitted to the downstream system at time t_{n+1}. Therefore,

$$\begin{aligned} & Z_{t_{n+1}}(TB, B_D + 1, B_T, T) - Z_{t_{n+1}}(TB, B_D, B_T, T) \\ = \;& Z_{t_n}(TB, B_D + 1, B_T, T) - Z_{t_n}(TB, B_D, B_T, T), \end{aligned}$$

so that (A.3) holds for $n + 1$.

Case 2: A cell arrives at time t_{n+1}.

If the cell is accepted by both systems, or if the cell is not accepted by either of the systems, then the queue lengths Z_t are either increased by one in both systems or unchanged in both systems, so that (A.3) trivially holds for $n+1$. If, however, the cell is accepted by only one of the control schemes. Then, the inductive assumption (A.3) implies that the cell is accepted by system $(TB, B_D + 1, B_T, T)$ and that

$$Z_{t_n}(TB, B_D + 1, B_T, T) = Z_{t_n}(TB, B_D, B_T, T) = B_D.$$

Therefore,

$$\begin{aligned} & Z_{t_{n+1}}(TB, B_D + 1, B_T, T) - Z_{t_{n+1}}(TB, B_D, B_T, T) \\ = \;& Z_{t_n}(TB, B_D + 1, B_T, T) + 1 - Z_{t_n}(TB, B_D, B_T, T) = 1, \end{aligned}$$

so that (A.3) holds for $n+1$.

Case 3: A token arrives at time t_{n+1}.

If the token is accepted by both systems, or if the token is not accepted by either of the systems, then the queue lengths Z_t are either decreased by one in both systems or unchanged in both systems, so that (A.3) trivially holds for $n+1$. If, however, the token is accepted by only one of the control schemes. Then, the inductive assumption (A.3) implies that the token is accepted by system (TB, B_D+1, B_T, T) and that

$$Z_{t_n}(TB, B_D+1, B_T, T) = Z_{t_n}(TB, B_D, B_T, T) + 1 = -B_T + 1.$$

Therefore,

$$\begin{aligned} & Z_{t_{n+1}}(TB, B_D+1, B_T, T) - Z_{t_{n+1}}(TB, B_D, B_T, T) \\ = \;& Z_{t_n}(TB, B_D+1, B_T, T) - 1 - Z_{t_n}(TB, B_D, B_T, T) = 0, \end{aligned}$$

so that (A.3) still holds for $n+1$.

Therefore, by induction, for all $n = 1, 2, \cdots$, relation (A.3) holds.

Let $K(TB, B_D+1, B_T, T) = \{n_1, n_2, \cdots, n_k, \cdots\}$, where $1 = n_1 < n_2 < \cdots < n_k < \cdots$.

Due to (A.3), for all $k = 1, 2, \cdots$,

$$Z_{d_{n_k}(TB,B_D+1,B_T,T)}(TB, B_D, B_T, T) \leq Z_{d_{n_k}(TB,B_D+1,B_T,T)}(TB, B_D+1, B_T, T).$$

Thus, all cells $v_{n_k}(TB, B_D+1, B_T, T)$ that are accepted and transmitted instantaneously to the downstream system in the system (TB, B_D+1, B_T, T) are also accepted and transmitted instantaneously to the downstream system in the system (TB, B_D, B_T, T). Hence, for all $k = 1, 2, \cdots$, we can define index n'_k as the one satisfying relation

$$v_{n'_k}(TB, B_D, B_T, T) = v_{n_k}(TB, B_D+1, B_T, T), \qquad k = 1, 2, \cdots. \tag{A.4}$$

Note that since there is at most one cell arrival at any instant, the indices n'_k are uniquely defined. Note also that $n'_1 = 1$. By definition, these indices verify the following relation

$$d_{n'_k}(TB, B_D, B_T, T) = d_{n_k}(TB, B_D+1, B_T, T), \qquad k = 1, 2, \cdots. \tag{A.5}$$

Owing to (A.3), at any time, a cell is accepted in system (TB, B_D, B_T, T) only if it is accepted in system $(TB, B_D + 1, B_T, T)$. Thus,

$$n'_{k+1} - n'_k \leq n_{k+1} - n_k, \qquad k = 1, 2, \cdots. \tag{A.6}$$

Moreover, if one more cell is accepted in system (TB, B_D+1, B_T, T) than in system (TB, B_D, B_T, T) after time $d_{n_k}(TB, B_D + 1, B_T, T)$, then $Z_t(TB, B_D + 1, B_T, T)$ is greater than $Z_t(TB, B_D, B_T, T)$ by one from the instant of that cell acceptance until the instant when one token is rejected in (TB, B_D, B_T, T) after time $d_{n_k}(TB, B_D + 1, B_T, T)$. Thus, there is at most one more cell acceptance in system $(TB, B_D + 1, B_T, T)$ than in system (TB, B_D, B_T, T) during the time interval $(d_{n_k}(TB, B_D + 1, B_T, T), d_{n_{k+1}}(TB, B_D + 1, B_T, T))$. Hence,

$$n_{k+1} - n_k \leq n'_{k+1} - n'_k + 1, \qquad k = 1, 2, \cdots. \tag{A.7}$$

As a consequence of (A.3), (A.4), (A.5), (A.6) and (A.7), we obtain that for all $k = 1, 2, \cdots,$

$$d_{n'_k+i}(TB, B_D, B_T, T) \leq d_{n_k+i}(TB, B_D + 1, B_T, T), \qquad 1 \leq i < n'_{k+1} - n'_k. \tag{A.8}$$

Define the counting process $D'_t(TB, B_D + 1, B_T, T)$ as a thinning of the departure counting process $D_t(TB, B_D + 1, B_T, T)$ by deleting points $d_{n_{k+1}-1}(TB, B_D + 1, B_T, T)$, provided $n_{k+1} - n_k = n'_{k+1} - n'_k + 1$, $k = 1, 2, \cdots$. Let $m_k = n'_k$, $k = 1, 2, \cdots$. It is now immediate from (A.4), (A.5) and (A.8), together with Lemma 2.3 that relation (A.2) holds. ■

Lemma 4.3 *Assume $B_D \geq 0$ and $B_T \geq 1$. Let cell arrival sequence A be arbitrarily fixed. If $K(LB, B_D+1, B_T, T)$ is an infinite set, then, there exist a counting process $D'_t(LB, B_D+1, B_T, T)$, constructed by thinning the departure counting process $D_t(LB, B_D + 1, B_T, T)$, and an infinite sequence of integers $\{m_k\}_{k=1}^{\infty}$, $1 = m_1 < m_2 < \cdots < m_k < \cdots$, such that*

$$\delta'_{m_k}(LB, B_D + 1, B_T, T) \prec \delta_{m_k}(LB, B_D, B_T, T), \qquad k = 1, 2, \cdots, \tag{A.9}$$

where $\delta'(LB, B_D+1, B_T, T) = \{\delta'_n(LB, B_D+1, B_T, T)\}_{n=1}^{\infty}$ is the sequence of inter-departure times associated with $D'_t(LB, B_D + 1, B_T, T)$.

Proof. Note that in the two leaky bucket control schemes, the sequences of token arrivals are not the same in general. The proof is therefore a little more tedious.

We first show that for all $t \geq 0$,

$$0 \leq Z_t(LB, B_D + 1, B_T, T) - Z_t(LB, B_D, B_T, T) \leq 2. \quad \text{(A.10)}$$

For any time $t \geq 0$, let $h_t(LB, B_D + 1, B_T, T) > 0$ (resp. $h_t(LB, B_D, B_T, T) > 0$) be the length of the time interval between t and the first time epoch when a token is generated in system $(LB, B_D + 1, B_T, T)$ (resp. (LB, B_D, B_T, T)) after time t. Note that when $Z_t(LB, B_D + 1, B_T, T) = -B_T$, this token generation occurs T time units later than the first cell arrival after t.

For the sake of brevity, we assume, without loss of generality, that cells do not arrive at the same time as tokens. In fact, if a cell arrives at the same time as a token, we can consider it as if the cell arrives just prior to the token.

We describe the behavior of the two systems when given the same cell arrival sequence by a finite state machine (Figure 1). This machine contains six states which differ according to the relative values of $Z_t(LB, B_D, B_T, T)$, $Z_t(LB, B_D + 1, B_T, T)$, $h_t(LB, B_D, B_T, T)$, and $h_t(LB, B_D + 1, B_T, T)$. In Figure 1, Z_t and h_t refer to the variables in system $(LB, B_D + 1, B_T, T)$, and Z'_t and h'_t to the variables in system (LB, B_D, B_T, T).

The systems start in state (1). This state remains unchanged whenever a token arrives (in both systems), or a cell arrives (in both systems) and $Z_t < B_D$. The systems transit to state (2) when a cell arrives and $Z_t = B_D$. The systems remain in state (2) until $Z'_t = -B_T + 1$ and a token arrives, in which case, they enter state (3). The systems transit out of state (3) if a token arrives in system $(LB, B_D + 1, B_T, T)$. The transition is to state (1), provided $Z'_t = -B_T$, or (4), provided $Z'_t > -B_T$.

A transition occurs from state (4) to (3) whenever a token arrives in system (LB, B_D, B_T, T). If a cell arrives, no transition occurs if $Z'_t < B_D$, or a transition to (5) occurs if $Z'_t = B_D$. No transition occurs from state (5) unless a token arrives in system (LB, B_D, B_T, T), in which case, the transition is to (6). If a token arrives in system $(LB, B_D + 1, B_T, T)$ while the systems are in state (6) the systems either return to state (5), provided $Z'_t > -B_T$, or enter state (3), provided $Z'_t = -B_T$. If, however, a cell arrives, then the systems either stay in state (6), provided $Z'_t < B_D$, or enter state (3), provided $Z'_t = B_D$ (cell overflow in system (LB, B_D, B_T, T)). Figure 2 illustrates the behavior of the system in states (3)—(6).

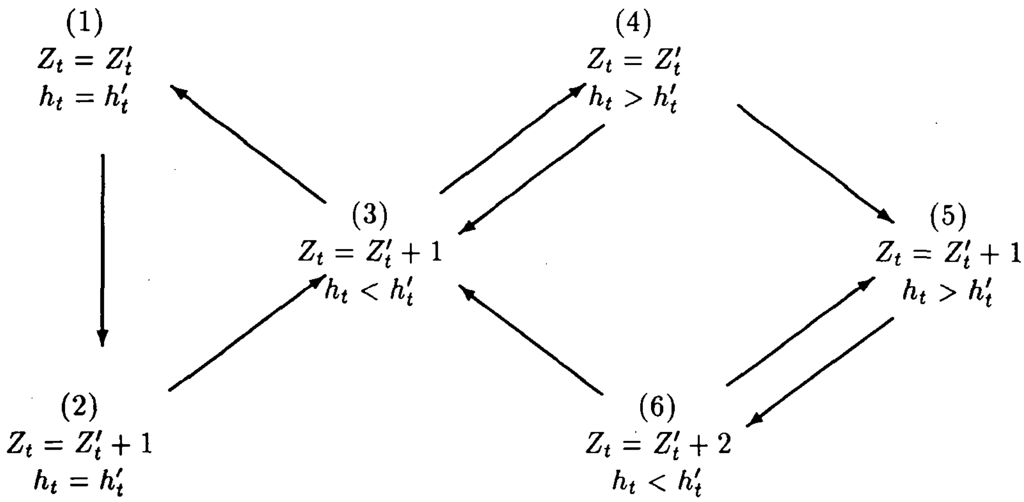

Figure 1: Finite state machine representation of two systems.

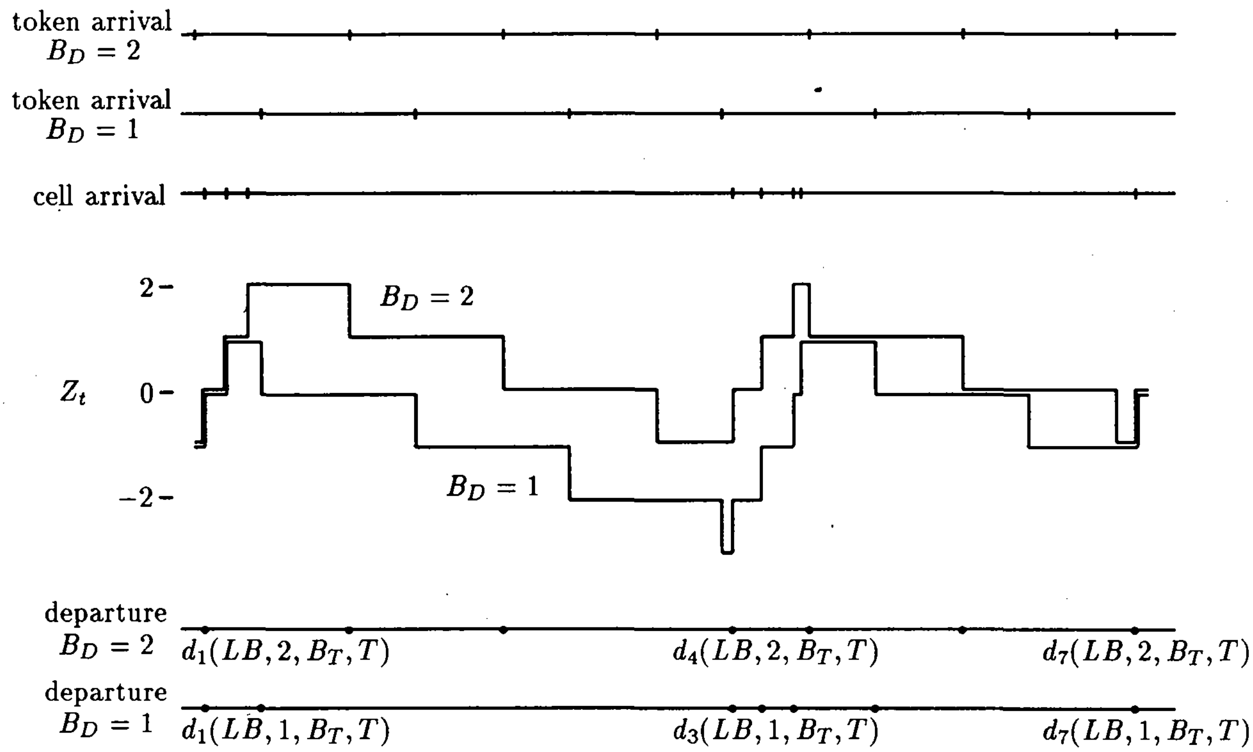

Figure 2: Example

An induction argument based on the above behavior establishes relation (A.10) for any time $t \geq 0$.

Let now $K(LB, B_D + 1, B_T, T) = \{n_1, n_2, \cdots, n_k, \cdots\}$, where $1 = n_1 < n_2 < \cdots < n_k < \cdots$.

Due to (A.10), for all $k = 1, 2, \cdots$,

$$Z_{d_{n_k}(LB,B_D+1,B_T,T)}(LB, B_D, B_T, T) \leq Z_{d_{n_k}(LB,B_D+1,B_T,T)}(LB, B_D + 1, B_T, T).$$

Thus, all cells $v_{n_k}(LB, B_D + 1, B_T, T)$ that are accepted and transmitted instantaneously to the downstream system in the system $(LB, B_D + 1, B_T, T)$ are also accepted and transmitted instantaneously to the downstream system in the system (LB, B_D, B_T, T). Hence, for all $k = 1, 2, \cdots$, we can define index n'_k as the one satisfying relation

$$v_{n'_k}(LB, B_D, B_T, T) = v_{n_k}(LB, B_D + 1, B_T, T), \qquad k = 1, 2, \cdots. \tag{A.11}$$

Note that the indices n'_k are uniquely defined, and that $n'_1 = 1$. By definition, these indices satisfy the following relation

$$d_{n'_k}(LB, B_D, B_T, T) = d_{n_k}(LB, B_D + 1, B_T, T), \qquad k = 1, 2, \cdots. \tag{A.12}$$

Henceforth, we refer to $d_{n_k}(LB, B_D + 1, B_T, T)$ as t_k, $k = 1, 2, \ldots$.

Due to (A.10), we have that

$$n'_{k+1} - n'_k - 1 \leq n_{k+1} - n_k \leq n'_{k+1} - n'_k + 1, \qquad k = 1, 2, \cdots. \tag{A.13}$$

It should be clear why $n_{k+1} - n_k$ can take values $n'_{k+1} - n'_k$ and $n'_{k+1} - n'_k + 1$. Since it is possible for system $(LB, B_D + 1, B_T, T)$ to have two additional cells in its data buffer beyond the number contained in system's (LB, B_D, B_T, T) (c.f., state (6)), the situation can arise where a cell arrives to find the data buffer full in $(LB, B_D + 1, B_T, T)$ but not in (LB, B_D, B_T, T) in the interval of time (t_k, t_{k+1}). Hence one less customer will depart during this time interval in system $(LB, B_D + 1, B_T, T)$.

Note that it can be shown that when $B_T = 1$, $0 \leq Z_t(LB, B_D + 1, 1, T) - Z_t(LB, B_D, 1, T) \leq 1$, so that $n'_{k+1} - n'_k \leq n_{k+1} - n_k \leq n'_{k+1} - n'_k + 1$.

Define set U_1 as

$$U_1 = \{k \mid n'_{k+1} - n'_k - 1 = n_{k+1} - n_k\}.$$

Note that U_1 may be finite or even empty. Let $U_1 = \{k_1, k_2, \cdots, k_r, \cdots, k_{r_0}\}$, where $1 < k_1 < k_2 < \cdots < k_r < \cdots$, $r_0 = 0$ if U_1 is empty, and $r_0 = \infty$ is U_1 is infinite. For convenience, we define $k_0 = 1$.

Observe that the equality $n'_{k+1} - n'_k - 1 = n_{k+1} - n_k$ holds if and only if

$$\begin{aligned} Z_{t_k}(LB, B_D + 1, B_T, T) &= Z_{t_k}(LB, B_D, B_T, T) + 2 \\ Z_{t_{k+1}}(LB, B_D + 1, B_T, T) &\leq Z_{t_{k+1}}(LB, B_D, B_T, T) + 1. \end{aligned}$$

One consequence of this property is that $k_{r+1} - k_r > 1$ for $r = 1, 2, \cdots$. Furthermore, there is some $k_r < k < k_{r+1}$ such that

$$Z_{t_k}(LB, B_D + 1, B_T, T) - Z_{t_k}(LB, B_D, B_T, T) < Z_{t_{k+1}}(LB, B_D + 1, B_T, T) - Z_{t_{k+1}}(LB, B_D, B_T, T).$$

As a consequence, there is an additional departure in system $(LB, B_D + 1, B_T, T)$ in (t_k, t_{k+1}), i.e., $n'_{k+1} - n'_k + 1 = n_{k+1} - n_k$. Let k'_r be the largest such integer, i.e.,

$$k'_r = \sup\{k | n'_{k+1} - n'_k + 1 = n_{k+1} - n_k; \quad k_r < k < k_{r+1}\}.$$

It then follows that for all $k'_r < k < k_r$, $n'_{k+1} - n'_k = n_{k+1} - n_k$. In Figure 2, $n_1 = 1$, $n_2 = 4$, $n_3 = 7$, $n'_1 = 1$, $n'_2 = 3$, $n'_3 = 7$, $k_1 = 2$, $k'_1 = 1$.

Let $U'_1 = \{k'_1, k'_2, \cdots, k'_r, \cdots, k'_{r_0}\}$. Define U_0 as

$$U_0 = \{k \mid n'_{k+1} - n'_k = n_{k+1} - n_k\}.$$

Let $U_2 = \{1, 2, \cdots, k, \cdots\} - U_0 - U_1 - U'_1$. By definition, $n'_{k+1} - n'_k + 1 = n_{k+1} - n_k$ for all $k \in U_2$.

Relations (A.10), (A.12), and (A.13), can now be used to yield, for all $k \in U_0 \cup U'_1 \cup U_2$,

$$d_{n'_k+i}(LB, B_D, B_T, T) \leq d_{n_k+i}(LB, B_D + 1, B_T, T), \qquad 1 \leq i < n'_{k+1} - n'_k. \tag{A.14}$$

It is easily seen (cf. states (5) and (6)) for all $k \in U_1$ that

$$d_{n'_k+i}(LB, B_D, B_T, T) \leq \left\{ \begin{array}{ll} d_{n_k+i}(LB, B_D + 1, B_T, T), & i = 1, \\ d_{n_k+i-1}(LB, B_D + 1, B_T, T), & 3 \leq i \leq n'_{k+1} - n'_k - 1. \end{array} \right\} \tag{A.15}$$

It follows from (A.14) and (A.15) together with Lemmas 2.7 and 2.3 that for all $1 \leq r \leq r_0$,

$$
\begin{aligned}
&\Big(\delta_{n_{k'_r}+1}(LB, B_D+1, B_T, T), \delta_{n_{k'_r}+2}(LB, B_D+1, B_T, T), \cdots, \delta_{n_{k'_r+1}}(LB, B_D+1, B_T, T), \\
&\delta_{n_{k_r}+1}(LB, B_D+1, B_T, T), \delta_{n_{k_r}+2}(LB, B_D+1, B_T, T), \cdots, \delta_{n_{k_r+1}}(LB, B_D+1, B_T, T)\Big) \\
&\quad \prec \quad \Big(\delta_{n'_{k'_r}+1}(LB, B_D, B_T, T), \delta'_{n_{k'_r}+2}(LB, B_D, B_T, T), \cdots, \delta_{n'_{k'_r+1}}(LB, B_D, B_T, T), \\
&\qquad \delta_{n'_{k_r}+1}(LB, B_D, B_T, T), \delta_{n'_{k_r}+2}(LB, B_D, B_T, T), \cdots, \delta_{n'_{k_r+1}}(LB, B_D, B_T, T)\Big)
\end{aligned}
\tag{A.16}
$$

Define counting process $D'_t(LB, B_D+1, B_T, T)$ as a thinning of the departure counting process $D_t(LB, B_D+1, B_T, T)$ by deleting points $d_{n_{k+1}-1}(LB, B_D+1, B_T, T)$ for all $k \in U_2$. The sequence of integers $\{m_r\}_{r=1}^{\infty}$ is defined as follows. If $r_0 = \infty$, then $m_1 = 1$, $m_{r+1} = n_{k_r}$, where $r = 1, 2, \cdots$. If $r_0 < \infty$, then $m_1 = 1$, $m_{r+1} = n_{r+k_{r_0}}$.

It is then follows from (A.11), (A.12) (A.14) and (A.16), together with Lemmas 2.7 and 2.3 that relation (A.9) holds. ∎

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